Permanent versus determinant: not via saturations

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Determinant Orbit Closure

- The determinant polynomial:

\[ \det_n := \sum_{\pi \in S_n} \text{sgn}(\pi) X_{1,\pi(1)} \cdots X_{n,\pi(n)} \in \text{Sym}^n \mathbb{C}^{n^2}, \]

where \( \text{Sym}^n \mathbb{C}^{n^2} \) is the space of homogeneous degree \( n \) forms in \( n^2 \) variables.

- We call \( \text{Sym}^n \mathbb{C}^{n^2} \) the ambient space.

- The group \( \text{GL}_{n^2} \) canonically acts on \( \text{Sym}^n \mathbb{C}^{n^2} \).

- Define the orbit closure

\[ \text{Det}_n := \overline{\text{GL}_{n^2} \det_n} := \{ g\det_n \mid g \in \text{GL}_{n^2} \}. \]

- In geometric complexity theory we search for functions in the vanishing ideal \( I(\text{Det}_n) \).

- These functions can be used to prove complexity lower bounds on polynomials like the permanent polynomial.
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Plethysms

- In each degree $d$ the coordinate ring of the ambient space splits:

\[ \mathcal{O}(\text{Sym}^n \mathbb{C}^{n^2})_d = \mathcal{O}(\text{Det}_n)_d \oplus I(\text{Det}_n)_d. \]

- The group $\text{GL}_{n^2}$ acts linearly on $\mathcal{O}(\text{Sym}^n \mathbb{C}^{n^2})_d$, on $\mathcal{O}(\text{Det}_n)_d$, and on $I(\text{Det}_n)_d$:

  Define $g \cdot f$ via $(g \cdot f)(x) := f(g^{-1}x)$.

- Therefore $\mathcal{O}(\text{Sym}^n \mathbb{C}^{n^2})_d$, $\mathcal{O}(\text{Det}_n)_d$, and $I(\text{Det}_n)_d$ are finite dimensional $\text{GL}_{n^2}$-representations.

- The irreducible representations of $\text{GL}_{n^2}$ that can occur in $\mathcal{O}(\text{Sym}^n \mathbb{C}^{n^2})_d$ are indexed by partitions with $dn$ boxes. For example, $(13, 13, 2, 2, 2, 2, 2)$ occurs in $\mathcal{O}(\text{Sym}^3 \mathbb{C}^9)_{12}$.

- To find a function in $I(\text{Det}_n)$ it is sufficient to find an irreducible representation that occurs in $\mathcal{O}(\text{Sym}^n \mathbb{C}^{n^2})$ but not in $\mathcal{O}(\text{Det}_n)$. This is Mulmuley & Sohoni’s approach using occurrence obstructions.

\[ S(\text{Det}_n) := \{ \text{partition } \lambda \mid \text{type } \lambda \text{ occurs in } \mathcal{O}(\text{Det}_n) \}, \]
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We search for \( \lambda \) in \( S(\text{Sym}^n \mathbb{C}^n^2) \setminus S(\text{Det}_n) \).

- Both sets are finitely generated \textbf{monoids}.
- For a monoid \( S \) let \( A := S - S \) the group generated by \( S \).
  - \( S \) is called \textbf{saturated} if
  \[
  \forall x \in A \forall k \in \mathbb{N}_{>0} : \text{ } kx \in S \Rightarrow x \in S.
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- The \textbf{saturation} \( \text{Sat}(S) \) of a monoid \( S \) is the smallest saturated monoid that contains \( S \).
- The difference \( \text{Sat}(S) \setminus S \) is called the set of \textbf{holes} of \( S \).
- Let \( S(\text{Det}_n)_{\leq n} \) denote the submonoid where we take only partitions with at most \( n \) rows. This is a natural restriction in geometric complexity theory.

\[ \text{Our main contribution} \]

\[ \text{Sat}(S(\text{Det}_n)_{\leq n}) = \{ \lambda \mid \text{the number of boxes of } \lambda \text{ is divisible by } n \}. \]

- Conclusion: The approach with the saturation \( \text{Sat}(S(\text{Det}_n)) \) is too coarse.
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- The difference \( \text{Sat}(S) \setminus S \) is called the set of holes of \( S \).
- Let \( S(\text{Det}_n)_{\leq n} \) denote the submonoid where we take only partitions with at most \( n \) rows. This is a natural restriction in geometric complexity theory.

**Our main contribution**

\[ \text{Sat}(S(\text{Det}_n)_{\leq n}) = \{ \lambda \mid \text{the number of boxes of } \lambda \text{ is divisible by } n \} \]

- Conclusion: The approach with the saturation \( \text{Sat}(S(\text{Det}_n)) \) is too coarse. We have to look at the holes of \( S(\text{Det}_n) \).
\[ S(\text{Det}_n) := \{ \text{partition } \lambda \mid \text{type } \lambda \text{ occurs in } O(\text{Det}_n) \} \]
\[ \subseteq S(\text{Sym}^n \mathbb{C}^2) := \{ \text{partition } \lambda \mid \text{type } \lambda \text{ occurs in } O(\text{Sym}^n \mathbb{C}^2) \}. \]

We search for \( \lambda \) in \( S(\text{Sym}^n \mathbb{C}^2) \setminus S(\text{Det}_n) \).

- Both sets are finitely generated monoids.
- For a monoid \( S \) let \( A := S - S \) the group generated by \( S \).
  - \( S \) is called saturated if
    \[ \forall x \in A \forall k \in \mathbb{N}_{>0} : \ kx \in S \Rightarrow x \in S. \]
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**Thm (Kumar 2012)**

For $n$ even, provided the Alon-Tarsi conjecture holds for $n$, we have that $S(\text{Det}_n)_{\leq n}$ contains all partitions where each row length is divisible by $n$.

- This is a direct statement about $S(\text{Det}_n)$, not just its saturation.
- But: It is only concerned with stretched partitions.
- And needs the Alon-Tarsi conjecture.
- Kumar’s proof actually studies a subvariety: Let $\text{Ch}_n \subseteq \text{Det}_n$ be the orbit closure

\[ \text{Ch}_n := \overline{\text{GL}_n(\sum X_{1,1}X_{2,2}\cdots X_{n,n})}. \]

Since $\text{Ch}_n \subseteq \text{Det}_n$ it follows $S(\text{Ch}_n) \subseteq S(\text{Det}_n)$.
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Saturation and Normalization

- For a monoid $S$ let $A(S) := S - S$ the group generated by $S$.
- The cone $C_Q(S)$ of $S$ is defined as
  \[ C_Q(S) := \{ q\lambda \mid \lambda \in S, \ q \in \mathbb{Q}_{\geq 0} \}. \]

Key Lemma

\[ \text{Sat}(S) = A(S) \cap C_Q(S). \]

Key Proposition

Let $\widehat{Ch}_n$ be the normalization of $Ch_n$. Then

\[ \text{Sat}(S(Ch_n)) = \text{Sat}(S(\widehat{Ch}_n)). \]

- It remains to determine $A(S(\widehat{Ch}_n))$ and $C_Q(S(\widehat{Ch}_n))$ to obtain $\text{Sat}(S(\widehat{Ch}_n))$.
- $S(\widehat{Ch}_n)$ can be described in terms of positivity of plethysm coefficients!
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- $S(\widetilde{Ch}_n)$ can be described in terms of positivity of plethysm coefficients!
Proof Idea

- $S(\widetilde{\text{Ch}_n}) = \{ \lambda \mid \lambda \text{ occurs in } \text{Sym}^n(\text{Sym}^{\lfloor \lambda \rfloor/n} \mathbb{C}^n) \}.$
- $C_Q(S(\widetilde{\text{Ch}_n}))$ contains all partitions with at most $n$ rows (Bürgisser, Christandl, 2009).
- The result for $A(S(\widetilde{\text{Ch}_n}))$ relies on the explicit construction of partitions with positive plethysm coefficients. We use the duality $\bigwedge^k \text{Sym}^k V \leftrightarrow \text{Sym}^k \text{Sym}^k V$.

For every $k$: Two long rows and a long first column of length $k$. The first two rows are treated separately.

- Linear algebra: With integer linear combinations we obtain all required partitions (upper triangular system of linear equations with 1s on the diagonal).

Conclusion

- To make the approach with $S(\text{Det}_n)$ work we have to look at those $\lambda$ that are holes of $S(\text{Det}_n)$. 
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- \( S(\widehat{Ch}_n) = \{ \lambda \mid \lambda \text{ occurs in } \text{Sym}^n(\text{Sym}^{\lfloor \lambda \rfloor/n} \mathbb{C}^n) \} \).
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Holes

- Holes have an interesting structure, even for $\text{Ch}_3$.
- For concreteness we calculated all holes for $\text{Ch}_3$ up to degree 9.
- A family of holes:
  For $j, k \in \mathbb{N}$ the partition $\lambda = (7 + 4k + 3j, 3 + 4k, 2 + 4k)$ is in $S(\text{Sym}^3 \mathbb{C}^3) \setminus S(\text{Ch}_3)$. 
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Thank you for your attention.