Exact solutions in Structured Low-Rank Approximation

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Problem Statement

\( p, q, r \in \mathbb{N} \)

\( E \) a linear/affine subspace of \( p \times q \) matrices with real entries

\( M \) a \( p \times q \) matrix, \( \Lambda = (\lambda_{i,j}) \) a \( p \times q \) positive matrix

\[
\|M\|_{\Lambda} = \sqrt{\sum_{i,j} \lambda_{i,j} M_{i,j}^2}
\]
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Structured (and weighted) Low-Rank Approximation

Given \( U \in E \), compute a matrix \( M \in E \) such that

- \( \text{Rank}(M) \leq r \);
- \( \|U - M\|_\Lambda \) is minimum.
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“Behind every linear data modeling problem there is a (hidden) low-rank approximation problem: the model imposes relations on the data which render a matrix constructed from exact data rank deficient.”

Markovskiy, 08
Some applications in symbolic-numeric computations

- $E = \text{Sylvester matrices} \rightsquigarrow \text{univariate approximate GCD}$

\[\begin{bmatrix}
a_3 & 0 & b_2 & 0 & 0 \\
a_2 & a_3 & b_1 & b_2 & 0 \\
a_1 & a_2 & b_0 & b_1 & b_2 \\
a_0 & a_1 & 0 & b_0 & b_1 \\
0 & a_0 & 0 & 0 & b_0
\end{bmatrix}\]
Some applications in symbolic-numeric computations

- $E =$Sylvester matrices $\leadsto$ univariate approximate GCD
- $E =$Hankel matrices $\leadsto$ denoising, signal processing, tensors

$\begin{bmatrix}
a & b & c & d & e \\
b & c & d & e & f \\
c & d & e & f & g \\
d & e & f & g & h \\
e & f & g & h & i
\end{bmatrix}$
Some applications in symbolic-numeric computations

- $E = \text{Sylvester matrices} \leadsto \text{univariate approximate GCD}$
- $E = \text{Hankel matrices} \leadsto \text{denoising, signal processing, tensors}$
- $E = \text{affine coordinate spaces} \leadsto \text{matrix completion}$

$$
\begin{bmatrix}
3 & ? & ? & 5 & 5 \\
1 & 2 & 3 & 2 & ? \\
10 & 4 & ? & 9 & -4 \\
6 & ? & 3 & 9 & 10 \\
? & 5 & -2 & ? & 9
\end{bmatrix}
$$
Some applications in symbolic-numeric computations

- \( E = \text{Sylvester matrices} \leadsto \text{univariate approximate GCD} \)
- \( E = \text{Hankel matrices} \leadsto \text{denoising, signal processing, tensors} \)
- \( E = \text{affine coordinate spaces} \leadsto \text{matrix completion} \)
- \( E = \text{Ruppert matrices} \leadsto \text{multivariate factorization} \)

\[
\begin{bmatrix}
0 & -2 & -a & 0 & -2b & -d \\
-1 & 0 & c & -b & 0 & e \\
a & 2c & 0 & d & 2e & 0 \\
0 & 0 & 0 & 1 & a & c \\
0 & 0 & 0 & -b & -d & -e
\end{bmatrix}
\]

\( XY^2 + aXY + bY^2 + cX + dY + e \in \mathbb{C}[X, Y] \text{ factors } \iff \text{rank } \leq 4 \)
Several approaches to SLRA:

Structured Total Least Norm (Park, Kaltofen, Zhi), Alternating projections (Bolte, Cadzow, Condat, Lewis, Malick, Hirabayashi), Riemannian optimization (Absil, Amodei, Meyer, Vandereycken), Matrix Factorization (Ishteva, Usevich, Markovsky), Newton iteration (Liu, Schost, S.)...
Several **approaches** to **SLRA**:


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The **EDdegree**, algebraic degree of optimization of Euclidean distances on algebraic varieties:  
*Draisma/Horobet/Ottaviani/Sturmfels/Thomas’13*
State of the art

Several approaches to SLRA:

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Complexity of global polynomial optimization: Abril Bucero, Faugère, Greuet, Lasserre, Mourrain, Nie, Parrilo, Safey, Schost, S., Sturmfels,…

The EDdegree, algebraic degree of optimization of Euclidean distances on algebraic varieties:
Draisma/Horobet/Ottaviani/Sturmfels/Thomas’13

Goals:

- **Certified and global SLRA** using symbolic (Gröbner bases) and symbolic-numeric algorithms (homotopy continuation methods)
- a priori estimates of the “algebraic difficulty” of the problem
  \( \rightsquigarrow \) explicit formulas for **EDdegree** of SLRA
- **Applications**: low-rank tensor approximation from diffusion magnetic resonance imaging (T. Schultz), Hankel matrices, approximate GCD
A symbolic approach to SLRA

$p, q, r \in \mathbb{N}$, $E$ a linear/affine subspace of $p \times q$ matrices with real entries

$M$ a $p \times q$ matrix, $\Lambda$ a $p \times q$ positive matrix, $\|M\|_\Lambda = \sqrt{\sum_{i,j} \lambda_{i,j} M_{i,j}^2}$

$D_r$: variety of $p \times q$ matrices of rank at most $r$

The minimizers of SLRA are algebraic

Minimizing a polynomial function $M \mapsto \sum_{i,j} \lambda_{i,j} (U_{i,j} - M_{i,j})^2$ on an algebraic variety $D_r \cap E$

$\leadsto$ SLRA can be modeled by polynomial system solving

Many possible approaches: Gröbner bases, border bases, homotopy methods, resultants, triangular sets, geometric resolution, ...
A symbolic approach to SLRA

$p, q, r \in \mathbb{N}$, $E$ a **linear/affine subspace** of $p \times q$ matrices with real entries

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**First step**: model the problem as a polynomial system

$\leadsto$ Ideal vanishing on the **regular critical points**
A symbolic approach to SLRA

\( p, q, r \in \mathbb{N}, E \) a linear/affine subspace of \( p \times q \) matrices with real entries
\( M \) a \( p \times q \) matrix, \( \Lambda \) a \( p \times q \) positive matrix, \( \|M\|_\Lambda = \sqrt{\sum_{i,j} \lambda_{i,j} M_{i,j}^2} \)
\( \mathcal{D}_r \): variety of \( p \times q \) matrices of rank at most \( r \)

The minimizers of SLRA are algebraic

Minimizing a polynomial function \( M \mapsto \sum_{i,j} \lambda_{i,j} (U_{i,j} - M_{i,j})^2 \) on an algebraic variety \( \mathcal{D}_r \cap E \)
\( \leadsto \) SLRA can be modeled by polynomial system solving

Many possible approaches: Gröbner bases, border bases, homotopy methods, resultants, triangular sets, geometric resolution, …

First step: model the problem as a polynomial system
\( \leadsto \) Ideal vanishing on the regular critical points

Technical assumptions for this talk:

- Finitely-many complex critical points on the smooth locus of \( \mathcal{D}_r \cap E \).
- Minimum is reached on the smooth locus of \( \mathcal{D}_r \cap E \).
Weighted low-rank approximation of the $4 \times 4$ determinant

\[ D(x) \in \mathbb{Q}[x_{11}, \ldots, x_{44}]: \text{det. of the matrix} \]

\[
\begin{bmatrix}
    x_{1,1} & \cdots & x_{1,4} \\
    \vdots & \ddots & \vdots \\
    x_{4,1} & \cdots & x_{4,4}
\end{bmatrix}
\]

\[ U: 4 \times 4 \text{ matrix picked at random} \]
\[ \Lambda: \text{positive } 4 \times 4 \text{ matrix} \]

\[ D(x) = 0 \]

\[
\text{Rank} \begin{bmatrix}
    \frac{\partial D}{\partial x_{11}} & \ldots & \frac{\partial D}{\partial x_{44}} \\
    \lambda_{11}(x_{11} - u_{11}) & \ldots & \lambda_{44}(x_{44} - u_{44})
\end{bmatrix} \leq 1
\]

Timings with \texttt{FGB} (\texttt{Faugère}):

\[ \Lambda \text{ generic, over } \mathbb{Q}: > 1 \text{ day} \]
\[ \Lambda = 1, \text{ over } \mathbb{Q}: 0.3 \text{s} \]

Can we explain these timings and/or find a better modeling?
Weighted low-rank approximation of the $4 \times 4$ determinant

$D(\mathbf{x}) \in \mathbb{Q}[x_{11}, \ldots, x_{44}]$: det. of the matrix

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U: 4 \times 4 \text{ matrix picked at random} \\
\Lambda: \text{positive } 4 \times 4 \text{ matrix}
\]

\[
D(\mathbf{x}) = 0
\]

\[
[y \quad 1] \left[ \begin{array}{cccc}
\frac{\partial D}{\partial x_{11}} & \ldots & \frac{\partial D}{\partial x_{44}} \\
\lambda_{11}(x_{11} - u_{11}) & \ldots & \lambda_{44}(x_{44} - u_{44})
\end{array} \right] = \left[ \begin{array}{cccc}
0 & \ldots & 0
\end{array} \right]
\]

variables: $x_{11}, \ldots, x_{44}, y$. 17 equations.
Weighted low-rank approximation of the $4 \times 4$ determinant

$D(\mathbf{x}) \in \mathbb{Q}[x_{11}, \ldots, x_{44}]$: det. of the matrix

$$
\begin{bmatrix}
    x_{1,1} & \ldots & x_{1,4} \\
    \vdots & & \vdots \\
    x_{4,1} & \ldots & x_{4,4}
\end{bmatrix}
$$

$U$: $4 \times 4$ matrix picked at random

$\Lambda$: positive $4 \times 4$ matrix

$$
D(\mathbf{x}) = 0
$$

$$
\begin{bmatrix}
y & 1
\end{bmatrix}
\begin{bmatrix}
  \frac{\partial D}{\partial x_{11}} & \ldots & \frac{\partial D}{\partial x_{44}} \\
  \lambda_{11}(x_{11} - u_{11}) & \ldots & \lambda_{44}(x_{44} - u_{44})
\end{bmatrix}
= \begin{bmatrix} 0 & \ldots & 0 \end{bmatrix}
$$

variables: $x_{11}, \ldots, x_{44}, y$. 17 equations.

Timings with FGb (*Faugère*):

- $\Lambda$ generic, over $\mathbb{Q}$: $> 1$ day
- $\Lambda = 1$, over $\mathbb{Q}$: 0.3s
Weighted low-rank approximation of the $4 \times 4$ determinant

$D(x) \in \mathbb{Q}[x_{11}, \ldots, x_{44}]$: det. of the matrix

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\begin{bmatrix}
 x_{1,1} & \cdots & x_{1,4} \\
 \vdots & \ddots & \vdots \\
 x_{4,1} & \cdots & x_{4,4}
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$$

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$D(x) = 0$

$$
\begin{bmatrix}
 y & 1 \\
 \partial D/\partial x_{11} & \cdots & \partial D/\partial x_{44} \\
 \lambda_{11}(x_{11} - u_{11}) & \cdots & \lambda_{44}(x_{44} - u_{44})
\end{bmatrix} = \begin{bmatrix} 0 & \cdots & 0 \end{bmatrix}
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variables: $x_{11}, \ldots, x_{44}, y$. 17 equations.

Timings with FGb (Faugère):

- $\Lambda$ generic, over $\mathbb{Q}$: $> 1$ day
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Can we explain these timings and/or find a better modeling?
The Euclidean distance degree

Draisma/Horobet/Ottaviani/Sturmfels/Thomas 13

\( V \in \mathbb{C}^n \) an algebraic variety, \( u \in \mathbb{C}^n \) a generic point. The \( \text{EDdegree}_\Lambda \) of \( V \) is the number of complex critical points of the function

\[
\lambda_1(x_1 - u_1)^2 + \cdots + \lambda_n(x_n - u_n)^2
\]
on the smooth locus of \( V \).
The Euclidean distance degree

$V \in \mathbb{C}^n$ an algebraic variety, $u \in \mathbb{C}^n$ a generic point. The EDdegree $\Lambda$ of $V$ is the number of complex critical points of the function

$$\lambda_1(x_1 - u_1)^2 + \cdots + \lambda_n(x_n - u_n)^2$$

on the smooth locus of $V$.

EDdegree(ellipse) = 4.
The Euclidean distance degree

$V \in \mathbb{C}^n$ an algebraic variety, $u \in \mathbb{C}^n$ a generic point. The $\text{EDdegree}_V$ of $V$ is the number of complex critical points of the function

$$\lambda_1(x_1 - u_1)^2 + \cdots + \lambda_n(x_n - u_n)^2$$

on the smooth locus of $V$.

Solution of SLRA:

critical point of the distance function on a linear section of a determinantal variety $\mathcal{D}_r \cap E$.

$\text{EDdegree(ellipse)} = 4$. 
The **EDdegree** of a projective variety is bounded by the **sum of the degrees of its polar classes**. If the weight matrix is generic, then equality holds.

**Duality:**

\[ \mathcal{N}_V = \{ (x, v) : x \in V_{\text{smooth}}, v \in N_x V \}. \]
Duality

Proposition (Draisma/Horobet/Ottaviani/Sturmfels/Thomas)

The \textbf{ED}degree of a projective variety is bounded by the \textbf{sum of} the degrees of its \textbf{polar classes}. If the weight matrix is generic, then equality holds.

\textbf{Duality:}

\[ \mathcal{N}_V = \{ (x, v) : x \in V_{\text{smooth}}, v \in \mathcal{N}_x V \} . \]

\[ \pi_2 : \mathcal{N}_V \to \mathbb{C}^n \quad V^* = \text{Im}(\pi_2) \]

\[ (x, v) \mapsto v \]

\textbf{Rank} \( r \) matrices are dual to \textbf{corank} \( r \) matrices.
Proposition (Draisma/Horobet/Ottaviani/Sturmfels/Thomas)

The **EDdegree** of a projective variety is bounded by the **sum of the degrees of its polar classes**. If the weight matrix is generic, then equality holds.

**Duality:**

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\mathcal{N}_V = \{(x, v) : x \in V_{\text{smooth}}, v \in N_x V\}.
\]

\[
\pi_2 : \mathcal{N}_V \to \mathbb{C}^n \quad V^* = \text{Im}(\pi_2)
\]

\[
(x, v) \mapsto v
\]

**Rank-deficient** matrices are dual to **rank 1** matrices

\[\rightsquigarrow \text{Segre varieties.}\]
$D(x) \in \mathbb{Q}[x_{11}, \ldots, x_{44}]$: determinant of the matrix $(x_{ij})$

$U$: $4 \times 4$ matrix picked at random

$\Lambda$: positive $4 \times 4$ matrix

Projective dual to $\{D(x) = 0\}$: rank 1 matrices

$$\varphi : \mathbb{C}^3 \times \mathbb{C}^4 \rightarrow \mathbb{C}^{16}$$

$$\begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ 1 \end{bmatrix}, \begin{bmatrix} b_1 & b_2 & b_3 & b_4 \end{bmatrix} \mapsto \begin{bmatrix} a_1 b_1 & a_1 b_2 & a_1 b_3 & a_1 b_4 \\ a_2 b_1 & a_2 b_2 & a_2 b_3 & a_2 b_4 \\ a_3 b_1 & a_3 b_2 & a_3 b_3 & a_3 b_4 \\ b_1 & b_2 & b_3 & b_4 \end{bmatrix}$$
Back to the $4 \times 4$ determinant: duality

$D(x) \in \mathbb{Q}[x_{11}, \ldots, x_{44}]$: determinant of the matrix $(x_{ij})$

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Projective dual to $\{D(x) = 0\}$: rank 1 matrices

\[\varphi : \mathbb{C}^3 \times \mathbb{C}^4 \rightarrow \mathbb{C}^{16}\]
\[
\begin{bmatrix}
a_1 \\
a_2 \\
a_3 \\
1
\end{bmatrix}
, 
\begin{bmatrix}
b_1 & b_2 & b_3 & b_4
\end{bmatrix}
\rightarrow
\begin{bmatrix}
a_1 b_1 & a_1 b_2 & a_1 b_3 & a_1 b_4 \\
a_2 b_1 & a_2 b_2 & a_2 b_3 & a_2 b_4 \\
a_3 b_1 & a_3 b_2 & a_3 b_3 & a_3 b_4 \\
b_1 & b_2 & b_3 & b_4
\end{bmatrix}
\]

Dual optimization problem:

\[\nabla \| \varphi(a_1, a_2, a_3, b_1, b_2, b_3, b_4) - U' \|_{\Lambda'}^2 = 0\]

\[\lambda'_{ij} = 1/\lambda_{ij} \quad u'_{ij} = \lambda_{ij} u_{ij}\]
Timings with FGb (primal/dual):

- \( \Lambda \) generic, over \( \mathbb{Q} \): >1day/891s
- \( \Lambda = 1 \), over \( \mathbb{Q} \): 0.3s/0.2s

Explanation of the gap between timings:

\[ ED\text{degree}_1 = 4 \quad ED\text{degree}_{\text{gen}} = 284. \]

+ general polynomial modeling for SLRA.
Timings with FGb (primal/dual):

- $\Lambda$ generic, over $\mathbb{Q}$: >1day/891s
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Explanation of the gap between timings:

$$ED\text{degree}_1 = 4 \quad ED\text{degree}_{\text{gen}} = 284.$$ 

+ general polynomial modeling for SLRA.

Strong correlation between timings and EDdegree of the problem. A priori estimates of the EDdegree?
Generic $E$, Generic weights, corank 1

critical points of $\lambda_{1,1}(x_{1,1} - u_{1,1})^2 + \cdots + \lambda_{p,q}(x_{p,q} - u_{p,q})^2$
on $(\mathcal{D}_r \cap E)_{smooth}$. 
critical points of \( \lambda_{1,1}(x_{1,1} - u_{1,1})^2 + \cdots + \lambda_{p,q}(x_{p,q} - u_{p,q})^2 \)
on \((D_r \cap E)_{\text{smooth}}\).

**Proposition**

Let \( E \) be a generic codimension \( s \) linear space of \( p \times q \) matrices, and \( D_r \) be the variety of rank-deficient matrices. The generic EDdegree of \( D_r \cap E \) equals

\[
\delta_0 + \cdots + \delta_{pq-2-s}.
\]

where

\[
\delta_\ell = \sum_{k=\ell}^{p+q-2} (-1)^{p+q-k} \binom{k+1}{\ell+1} \nu_k
\]

\[
\nu_k = \left[ s^{p-1} t^{q-1} \right] (1 + s)^p (1 + t)^q (t + s)^k.
\]
critical points of \( \lambda_{1,1}(x_{1,1} - u_{1,1})^2 + \cdots + \lambda_{p,q}(x_{p,q} - u_{p,q})^2 \) on \((D_r \cap E)_{\text{smooth}}\).

**Proposition**

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\delta_{\ell} = \sum_{k=\ell}^{p+q-2} (-1)^{p+q-k} \binom{k+1}{\ell+1} v_k
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\[
v_k = \left[ s^{p-1} t^{q-1} \right] (1+s)^p (1+t)^q (t+s)^k.
\]

+ similar statement for rank 1 matrices by duality.
critical points of $\lambda_{1,1}(x_{1,1} - u_{1,1})^2 + \cdots + \lambda_{p,q}(x_{p,q} - u_{p,q})^2$
on $\quad$ on $(\mathcal{D}_r \cap E)_{\text{smooth}}.$

**Proposition**

Let $E$ be a **generic codimension $s$** linear space of $p \times q$ matrices, and $\mathcal{D}_r$ be the variety of **rank-deficient matrices**. The generic **ED degree** of $\mathcal{D}_r \cap E$ equals

$$\delta_0 + \cdots + \delta_{pq-2-s}.$$ 

where

$$\delta_\ell = \sum_{k=\ell}^{p+q-2} (-1)^{p+q-k} \binom{k+1}{\ell+1} v_k$$

and

$$v_k = \left[ s^{p-1} t^{q-1} \right] (1 + s)^p (1 + t)^q (t + s)^k.$$ 

+ similar statement for rank 1 matrices by duality.
Interchangeable with Schubert calculus, *Lascoux*.
EDdegree dramatically decreases.
EDdegree dramatically decreases.

Role of the isotropic quadric \( \sum x_{ij}^2 = 0 \):

**Conjecture**

Let \( r = \min(p, q) - 1 \) and \( Z \) be the locus of non-tranverse intersection between \( \mathcal{D}_r \cap E \) and the isotropic quadric.

\[
ED\text{degree}_1(\mathcal{D}_r \cap E) = ED\text{degree}_{\text{gen}}(\mathcal{D}_r \cap E) - ED\text{degree}_{\text{gen}}(Z).
\]
EDdegree dramatically decreases.

Role of the isotropic quadric $\sum x_{ij}^2 = 0$:

**Conjecture**

Let $r = \min(p, q) - 1$ and $Z$ be the locus of non-tranverse intersection between $D_r \cap E$ and the isotropic quadric.

$$EDdegree_1(D_r \cap E) = EDdegree_{\text{gen}}(D_r \cap E) - EDdegree_{\text{gen}}(Z).$$

+ explicit formula for $EDdegree_{\text{gen}}(Z)$. Tested on many examples.
Special linear subspaces: resultant and approximate GCD

\[ f(X) = f_m X^m + \cdots + f_1 X + f_0 \]
\[ g(X) = g_n X^n + \cdots + g_1 X + f_0 \]
\[ \|(f, g)\|^2 = \alpha_m f_m^2 + \cdots + \alpha_0 f_0^2 + \beta_n g_n^2 + \cdots + \beta_0 g_0^2 \]

\( V_k \subset \mathbb{P}^{m+n+1} \):

pairs of pols sharing a GCD of degree at least \( k \).
Special linear subspaces: resultant and approximate GCD

\[ f(X) = f_mX^m + \cdots + f_1X + f_0 \]
\[ g(X) = g_nX^n + \cdots + g_1X + f_0 \]
\[ \|(f, g)\|_2^2 = \alpha_m f_m^2 + \cdots + \alpha_0 f_0^2 + \beta_n g_n^2 + \cdots + \beta_0 g_0^2 \]

\[ V_k \subset \mathbb{P}^{m+n+1} : \]
  pairs of pols sharing a GCD of degree at least \( k \).

**Theorem**

The **generic EDdegree** of \( V_k \) equals that of the Segre variety of \( (k + 1) \times (n + m - 2k + 2) \) matrices of rank 1.

\( \leadsto \) Sylvester matrices
Conclusion

Algebraic geometry techniques: analysis of singularities, characteristic class computations

 Computational aspects, complexity of SLRA.
Conclusion

Algebraic geometry techniques: analysis of singularities, characteristic class computations

Computational aspects, complexity of SLRA.

Hankel matrices, some LRA of tensors
**Conclusion**

*Algebraic geometry* techniques: analysis of singularities, characteristic class computations

\[\uparrow\]

**Computational aspects, complexity of SLRA.**

Hankel matrices, some LRA of tensors
\[\mapsto\] rank 2 approx of a symmetric $3 \times 3 \times 3 \times 3$ from biomedical imaging. EDdegree: 195.

- **Homotopy continuation**, Bertini: 2 hours (precomp) + 1 min (instance)
- **Gröbner bases**, FGb: 11 min
- **Convex relaxation**, B. Mourrain, M.A. Bucero: 1 min
- conjecture for the formula of the EDdegree of SLRA for non-generic weights?
- Algos: exploiting duality for SLRA?
- Number of real critical points?
- Algos with provable polynomial complexity in the algebraic degree?

⇝ Mohab Safey El Din’s talk
- conjecture for the formula of the EDdegree of SLRA for non-generic weights?
- Algos: exploiting duality for SLRA?
- Number of real critical points?
- Algos with provable polynomial complexity in the algebraic degree?

~Mohab Safey El Din’s talk

Thank you!