Gauss maps of Toric varieties

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$k = \overline{k}, \ p = \text{char } k \geq 0$

$X \subset \mathbb{P}^N : n$-dim. projective variety / $k$

$G(n, \mathbb{P}^N) := \left\{ W \subset \mathbb{P}^N \mid W : n$-dim. linear subvar. $\right\}$

$= G(n + 1, k^{N+1})$

**Definition**

The Gauss map $\gamma$ of $X$ is the rational map

$$\gamma = \gamma_X : X \dashrightarrow G(n, \mathbb{P}^N) : x \mapsto T_xX,$$

where $T_xX \subset \mathbb{P}^N$ is the embedded tangent space of $X$ at a smooth point $x \in X$. 
Example

$X \subset \mathbb{P}^N : \text{linear}

\Rightarrow T_xX = X \text{ for } \forall x \in X, \text{ i.e. } \gamma_X : \text{const. map.}

In fact, $\gamma_X$ is a const. map $\iff X \subset \mathbb{P}^N$ is linear.

Example

$p = 0, \ C \subset \mathbb{P}^N : \text{curve, not linear } \Rightarrow \gamma_C$ is birational.

Geometrically, a gen. tangent line of $C$ is tangent to $C$ at only one point.

In $p = 0$, a gen. fiber of any Gauss maps is linear.
In $p > 0$, for any variety $F \subset \mathbb{P}^{N'}, \ \exists X \subset \mathbb{P}^N$

s.t. a gen. fiber (with. red. structure) $\simeq F$ [Fukasawa].
Gauss map is a close notion to dual varieties, 

\[
X^* = \{H \in \mathbb{P}^{N^\vee} = \mathbb{G}(N - 1, \mathbb{P}^N) \mid H : \text{tangent to } X\},
\]

\[
\gamma_X(X) = \{W \in \mathbb{G}(n, \mathbb{P}^N) \mid W : \text{tangent to } X\}.
\]

Dual varieties of toric varieties are studied by many people. For example, Di Rocco determined when smooth toric varieties have positive dual defect (i.e. \(N - 1 - \dim X^* > 0\)).

On the other hand, Gauss maps of toric varieties are not studied so much yet.
$M \cong \mathbb{Z}^n$: abelian group

$k[M] := \bigoplus_{u \in M} k z^u$: simigroup ring

$T_M := \text{Spec } k[M] \cong (k^\times)^n$: algebraic torus

**Definition**

For a finite set $A = \{u_0, \ldots, u_N\} \subset M$, we define a (not necessarily normal) toric variety $X_A$ as

$$X_A := \overline{\varphi_A(T_M)} \subset \mathbb{P}^N$$

for

$$\varphi_A : T_M \to \mathbb{P}^N : t \mapsto [z^{u_0}(t) : z^{u_1}(t) : \cdots : z^{u_N}(t)].$$
Example

\[ A = \{0, e_1, \ldots, e_n\} \text{ for a basis } e_1, \ldots, e_n \in M \Rightarrow X_A = \mathbb{P}^n. \]
\[ A = \{0, e_1, e_2, e_1 + e_2\} \Rightarrow X_A = \mathbb{P}^1 \times \mathbb{P}^1 \subset \mathbb{P}^3. \]

Definition

For \( A = \{u_0, \ldots, u_N\} \subset M \), we set

\[ B = B_A := \{u_{i_0} + u_{i_1} + \cdots + u_{i_n}\} \subset M, \]

where we take the sum for \( n + 1 \) elements
\( u_{i_0}, u_{i_1}, \ldots, u_{i_n} \in A \) which spans \( M_k := M \otimes_{\mathbb{Z}} k \simeq k^n \) as a \( k \)-affine space.
Example

\[ A = \{(0, 0), (0, 1), (1, -1), (-1, -1)\} \subset \mathbb{Z}^2 \]

char \(k \neq 2\)  \hspace{1cm} \text{char} \, k = 2

\[ A \times B \]

\[ \bullet \]

\[ \times \]
For a finite subset $A \subset M$ and $B = B_A$ as above, we consider the projection $\pi$

$$\pi : M \to M/(\langle B - B \rangle_R \cap M),$$

where $\langle B - B \rangle_R \subset M_R = M \otimes \mathbb{R}$ is the $\mathbb{R}$-vector space generated by $B - B := \{u - u' \mid u, u' \in B\}$.

Roughly speaking, $\pi$ is the projection which contracts $B$ to a point.
Example

\[ A = \{(0, 0), (0, 1), (1, -1), (-1, -1)\}, \text{ char } k = 2, \]
\[ B = \{(1, 0), (-1, 0)\}, \]
\[ \langle B - B \rangle_{\mathbb{R}} = \mathbb{R}(2, 0) = \mathbb{R}(1, 0), \quad \langle B - B \rangle_{\mathbb{R}} \cap \mathbb{Z}^2 = \mathbb{Z}(1, 0) \]

\[ \pi : \mathbb{Z}^2 \to \mathbb{Z}^2 / \mathbb{Z}(1, 0) \simeq \mathbb{Z} \]
Theorem (Furukawa - I)

$k, M, A, B = B_A$ as above.
Assume $A$ spans $M$ as a $\mathbb{Z}$-affine space.
Then the Gauss map $\gamma : X_A \rightarrow \mathbb{G}(n, \mathbb{P}^N)$ of $X_A$ can be identified with the rational map $X_A \rightarrow X_B$ induced by the natural inclusion $\langle B - B \rangle \hookrightarrow M$, that is,

- $\gamma(X_A) \simeq X_B$,
- the restriction of $\gamma : X_A \rightarrow \gamma(X_A) \simeq X_B$ on $T_M \subset X_A$ is $T_M = \text{Spec } k[M] \rightarrow \text{Spec } k[\langle B - B \rangle] \subset X_B$ induced by $\langle B - B \rangle \subset M$,
- each irreducible component of a general fiber of $\gamma$ (with red. structure) $\simeq X_{\pi(A)}$. 
Sketch of the proof

Since we have a parametrization

\[ \varphi_A : T_M \simeq (k^\times)^n \to X_A \subset \mathbb{P}^N : t \mapsto [t^{u_0} : \cdots : t^{u_N}] \],

we can compute the tangent space \( T_{\varphi(t)}X_A \), which is spanned by \( n + 1 \) row vectors of the \( n + 1 \times N + 1 \) matrix

\[
\begin{bmatrix}
t^{u_0} \cdot \begin{bmatrix} 1 \\ u_0 \end{bmatrix} & t^{u_1} \cdot \begin{bmatrix} 1 \\ u_1 \end{bmatrix} & \cdots & t^{u_N} \cdot \begin{bmatrix} 1 \\ u_N \end{bmatrix}
\end{bmatrix}.
\]

Since \( \mathbb{G}(n, \mathbb{P}^N) \hookrightarrow \mathbb{P}\left( \bigwedge^{n+1} k^{N+1} \right) \), we take \( n + 1 \) minors of the matrix and we obtain the theorem.
Example

\[ A = \{(0, 0), (0, 1), (1, -1), (-1, -1)\} \subset \mathbb{Z}^2, \text{ char } k = 2, \]
\[ \langle B - B \rangle = \mathbb{Z}(2, 0) \subset M = \mathbb{Z}^2, \]

\[ \gamma(X_A) \simeq X_B = \mathbb{P}^1 \text{ (line)}, \]

Hence a general fiber (with red. structure) \( \simeq X_{\pi(A)} \) : conic.
Corollary

Let $\text{char } k = 0$, $\delta := \dim X_A - \dim \gamma(X_A)$.

Then there exist disjoint torus invariant closed subvarieties $X_0, \ldots, X_\delta$ of $X_A$ such that $X_A$ is the join of $X_0, \ldots, X_\delta$.

Corollary

Let $\text{char } k > 0$, $A' \subset \mathbb{Z}^{n'}$, $A'' \subset \mathbb{Z}^{n''}$.

Then there exists a toric variety $X_A \subset \mathbb{P}^N$ such that

- $\gamma(X_A) \simeq X_{A'}$,

- each irreducible component of a general fiber of $\gamma$ (with red. structure) $\simeq X_{A''}$.
Remark

Above theorem can be generalized to higher order Gauss maps [Di Rocco-Jabbusch-Lundman].

Thank you!