Multiplicities of eigenvalues of tensors

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linear algebra

- $A$: $n \times n$ matrix
- $\lambda$: eigenvalue of $A$
- $am(\lambda)$: algebraic multiplicity
- multiplicity of $\lambda$ in characteristic polynomial
- $gm(\lambda)$: geometric multiplicity
- dimension of eigenspace of $\lambda$
- $am(\lambda) \geq gm(\lambda)$
- $A$ is a tensor in $\mathbb{C}^n \otimes \mathbb{C}^n$
- how about tensors in $(\mathbb{C}^n)^\otimes m$?
spectral (hyper-)graph theory

- $G$: graph
- $A$: adjacency matrix of a graph $G$
- $D$: diagonal matrix of degrees
- $\#\text{connected components} = \text{am}(0)$ of $D - A$
- $\#\text{bipartite connected components} = \text{am}(0)$ of $D + A$
- how about hyper-graphs?
- much more complicated
notations

- $\mathcal{T} = (t_{ji_2\ldots i_m})$: tensor (hyper-matrix) in $(\mathbb{C}^n)^{\otimes m}$
- $x = (x_1, \ldots, x_n)$ variables in $\mathbb{C}^n$

$$(\mathcal{T}x^{m-1})_j = \sum_{i_2,\ldots,i_m = 1}^{n} t_{ji_2\ldots i_m}x_{i_2}\cdots x_{i_m}, j = 1, 2, \ldots, n$$

- $\mathcal{T}x^{m-1}$ gives a system of $n$ homogeneous polynomial equations of degree $(m-1)$ in $n$ variables
- $x^{[m-1]} = (x_1^{m-1}, x_2^{m-1}, \ldots, x_n^{m-1})^T$
eigenvalue

Let $\mathcal{T}$ be a tensor in $(\mathbb{C}^n)^\otimes m$ and $\lambda$ be a complex number. If

$$\mathcal{T}x^{m-1} = \lambda x^{[m-1]}$$

has a nonzero solution, then $\lambda$ is called an eigenvalue of $\mathcal{T}$ and the solution space $V(\lambda)$ is called the eigen-variety of $\lambda$. 

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**eigen-equation**
Example

\[ T = (t_{ijk}) \in (\mathbb{C}^2)^\otimes 3 \] with

\[ t_{111} = 2, \quad t_{122} = t_{222} = 1, \quad \text{other } t_{ijk} = 0. \]

eigen-equation

\[ 2x_1^2 + x_2^2 = \lambda x_1^2, \quad x_2^2 = \lambda x_2^2. \]

\[ \lambda = 1 \] is an eigenvalue and the eigen-variety is the union of two lines

\[ V(1) = \mathbb{C}(\sqrt{-1}, 1) \cup \mathbb{C}(-\sqrt{-1}, 1). \]
motivation definition conjecture and evidence

**multiplicities**

- eigen-equation $T x^{m-1} - \lambda x^{[m-1]}$: $n$ homogeneous polynomials of degree $(m - 1)$ in $n$ variables.
- well-known: it has the resultant $\chi(\lambda)$

**Definition**

Let $T \in (\mathbb{C}^n)^m$ be a tensor and let $\lambda_0$ be an eigenvalue of $T$. The polynomial $\chi(\lambda)$ is called the **characteristic polynomial** of $T$. The multiplicity of $\lambda_0$ in $\chi(\lambda)$ is called the **algebraic multiplicity** of $\lambda_0$ and it is denoted by $am(\lambda_0)$. The dimension of the eigen-variety $V(\lambda_0)$ is called the **geometric multiplicity** of $\lambda_0$ and it is denoted by $gm(\lambda_0)$.
\( \lambda_0 \) is an eigenvalue of \( T \) iff \( \chi(\lambda_0) = 0 \)

\#eigenvalues = \( n(m-1)^{n-1} \) (counting with multiplicities)

\( \lambda, \mu \) eigenvalues: \( V(\lambda) \cap V(\mu) = \{0\} \) iff \( \lambda \neq \mu \)

\( gm_\mathbb{R}(\lambda) \leq gm_\mathbb{C}(\lambda) \) (equality always holds in matrix case)

\( \mathbb{O}(n) \): orthogonal group

\( \mathbb{O}(n) \) acts on \( (\mathbb{C}^n)^\otimes m \) (induced by the action on \( \mathbb{C}^n \))

\( gm(0) \) is invariant under the action of \( \mathbb{O}(n) \)

\( am(\lambda) \) is not invariant under the action of \( \mathbb{O}(n) \) (it is invariant in matrix case)

example is coming soon
inequality

- matrix case: \( am(\lambda) \geq gm(\lambda) \)
- expect: similar inequality for tensors
- our guess:

\[
am(\lambda) \geq gm(\lambda)(m - 1)^{gm(\lambda) - 1}
\]
trivial example: identity tensor

- $\mathcal{I}$: identity tensor ($t_{i...i} = 1$, $t_{i_1,...,i_m} = 0$ all others)
- $\mathcal{T} = \lambda_0 \mathcal{I}$, $\lambda_0 \in \mathbb{C}$
- $\mathcal{T}$ has the unique eigenvalue $\lambda_0$
- $\text{am}(\lambda_0) = n(m - 1)^{n-1}$
- $\text{gm}(\lambda_0) = n$ (eigen-variety is the whose space)
- equality holds
list of evidence

- matrix case \((m = 2)\)
- generic tensors
- numeric examples
- eigen-variety is a coordinate space
- small marginal rank symmetric tensors
**Theorem (K.Y–S.L.H 2015)**

Let $T \in (\mathbb{C}^n)^\otimes m$ be generic then

$$am(\lambda) = gm(\lambda) = 1$$

for every eigenvalue $\lambda$ of $T$. 
A = (a_{ijk}) \in (\mathbb{C}^2)^{\otimes 3} \text{ be the tensor given by } a_{112} = 1, a_{ijk} = 0 \text{ for all other } i, j, k

A \text{ has the unique eigenvalue } 0 \text{ so } am(0) = 4 \text{ and } gm(0) = 1.

apply \quad P = \begin{pmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ -1/\sqrt{2} & -1/\sqrt{2} \end{pmatrix} \text{ to } A \text{ to obtain a new tensor } B

B \text{ has eigenvalues } 0 \text{ and } -1/\sqrt{2}

am(0) = 2 \text{ and } gm(0) = 1 \text{ (am(0) is not invariant)}

am\left(-\frac{1}{\sqrt{2}}\right) = 2 \text{ and } gm\left(-\frac{1}{\sqrt{2}}\right) = 1
coordinate case


Let $\mathcal{T} \in (\mathbb{C}^n)^\otimes m$ be a tensor and let $\lambda_0$ be an eigenvalue of $\mathcal{T}$. Assume that

$$P \cdot \{ x \in \mathbb{C}^n \mid x_{g\text{m}(\lambda_0)+1} = \cdots = x_n = 0 \}$$

is the eigen-variety of $\lambda_0$ for some permutation matrix $P \in \mathfrak{S}_n$. Then

$$a\text{m}(\lambda_0) \geq g\text{m}(\lambda_0)(m - 1)^{g\text{m}(\lambda_0) - 1}.$$
lower marginal rank symmetric tensors

- $\mathcal{A} \in S^m \mathbb{C}^n$: symmetric tensor
- view $\mathcal{A}$ as a linear map $L_{\mathcal{A}} : S^{m-1} \mathbb{C}^n \rightarrow \mathbb{C}^n$
- the marginal rank of $\mathcal{A}$: $\text{mrank}(\mathcal{A}) = \text{rank}(L_{\mathcal{A}})$
- $M(s, n)$: variety of all symmetric tensors with marginal rank at most $s$
lower marginal rank symmetric tensors


Assume that $s \leq n$ and $A \in M(s, n)$ is a generic element then

$$\text{gm}(0) = n - s.$$ 


Assume that $s \leq n$ and $A \in M(s, n)$ is a generic element then

$$\text{am}(0) \geq (n - s)(m - 1)^{n-1}.$$ 

Hence we have

$$\text{am}(0) \geq \text{gm}(0)(m - 1)^{\text{gm}(0)-1}.$$
more to consider

- \( \text{am}(\lambda) \geq \text{gm}(\lambda)(m - 1)^{\text{gm}(\lambda) - 1} \) involves only dimension of \( V(\lambda) \)
- number of irreducible components, degrees of components?
- scheme structure of \( V(\lambda) \)
may expect equality (intersection theory)

evidence:

**Theorem (Fulton)**

Let $V_1, \ldots, V_n$ be hypersurfaces in $\mathbb{C} \times \mathbb{P}^n$ defined by

$$T x^{m-1} - \lambda x^{[m-1]}$$

and assume that $\cap V_i$ is finite then $am(\lambda)$ can be interpreted as

$$am(\lambda_0) = \sum_{p} i(p, V_1 \cdots V_n; \mathbb{P}^n),$$

where $p$ runs through all points in $\cap V_i$ corresponding to $\lambda_0$ and $i(p, V_1 \cdots V_n; \mathbb{P}^n)$ is the intersection multiplicity of $V_1, \cdots, V_n$ at $p$. 

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Intersection Theory
Thank You For Your Attention !