A lifted square formulation for certifiable Schubert calculus

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Schubert calculus

- Regards linear spaces incident to fixed linear spaces
  
  **Example** The problem of four lines

- Grassmannian $\text{Gr}(a, n)$: set of $a$-dim linear subspaces of $C^n$

- **Schubert condition** $w$: list of integers
  
  $$1 \leq w_1 < w_2 < \cdots < w_a \leq n$$

- **(Complete) flag** $F$: list of nested linear subspaces
  
  $$0 \subsetneq F_1 \subsetneq F_2 \subsetneq \cdots \subsetneq F_n = C^n$$
Schubert problem

- **Schubert variety** $X_w F_\bullet$:

$$X_w F_\bullet := \{ x \in \text{Gr}(a, n) \mid \dim x \cap F_{w_i} \geq i, \text{ for } i = 1, \ldots, a \}.$$ 

- $|w| := \text{codim}_{\text{Gr}(a, n)} X_w F_\bullet$.

- A **Schubert problem** is a list of Schubert conditions $(w^1, \ldots, w^\ell)$ such that

$$|w^1| + \cdots + |w^\ell| = \dim(\text{Gr}(a, n)).$$

Example ((2, 4), (2, 4), (2, 4), (2, 4)) is a Schubert problem in $\text{Gr}(2, 4)$ as

$$4|2, 4| = 4(1) = 4 = \dim(\text{Gr}(2, 4))$$
Schubert problem instance

- **Instance Schubert problem** \((w^1, \ldots, w^\ell)\): an intersection
  \[ X_{w^1}F^1 \cap \cdots \cap X_{w^\ell}F^\ell \]
  with flags \(F^i\) in general position.

**Example** The problem of four lines is an instance
\[ X_{(2,4)}F^1 \cap X_{(2,4)}F^2 \cap X_{(2,4)}F^3 \cap X_{(2,4)}F^4 \]

- Schubert problems are traditionally formulated using Plücker coordinates or (local) Stiefel coordinates.
- These usually involve more equations than variables.
- Bad for numerical methods/prevents certification.
• There is a primal-dual square formulation for Schubert problems. [Hauenstein-H-Sottile]

• Thus we may use numerical methods to solve and certify instances of Schubert problems, so we may study problems which are infeasible by symbolic methods.

• **Drawback**: the formulation uses a lot of extra variables.

• **Goal**: a more efficient square formulation (fewer variables)

• **Stiefel coordinates $\mathcal{X}$**: set of $n \times a$ complex matrices with linearly independent columns $e_i$ such that

$$\mathcal{X} \ni x \mapsto \langle e_1(x), \ldots, e_a(x) \rangle$$

gives a bijection onto an open set of $X_w F_\bullet$. 
To achieve a square formulation, we “lift” a Schubert condition on \( \text{Gr}(a, n) \) to a geometric condition in a flag variety \( F_{\ell}(1, \ldots, a; n) \).

**Example** \( \text{Gr}(3, 8), w = (3, 5, 8) \): \( x \in X_w F \) must satisfy three conditions

- \( \dim x \cap F_3 \geq 1 \),
- \( \dim x \cap F_5 \geq 2 \), and
- \( \dim x \cap F_8 \geq 3 \) (trivial).

Thus, given Stiefel coordinates \( \mathcal{X} \) for \( X_w F \), there are numbers \( \alpha_{1,2}, \alpha_{1,3}, \alpha_{2,3} \) such that

\[
e_1 + \alpha_{1,2} e_2 + \alpha_{1,3} e_3 \in F_3,
\]

\[
e_2 + \alpha_{2,3} e_3 \in F_5, \text{ and}
\]

\[
e_3 \in F_8.
\]
Lifted formulation for $X_w F\bullet \in \text{Gr}(a, n)$

- Choose Stiefel coordinates $\mathcal{X}$ for $\text{Gr}(a, n)$.

$$\mathcal{X} \ni x \mapsto \langle e_1(x), \ldots, e_a(x) \rangle$$

- Introduce lifting coordinates $\{\alpha_{k,i}\}$ and vectors $g_k(x)$ for $k \leq a$:

$$g_k(x) := e_k + \sum_{k<i \leq a} \alpha_{k,i} e_i(x).$$

- Choose linear forms $f_1, \ldots, f_n$ such that $F_j$ is defined by the vanishing of $f_{j+1}, \ldots, f_n$. The equations for $X_w F\bullet$ are

$$f_j(g_i(x)) = 0 \quad \text{for} \quad i \leq a, \ j > w_i.$$
Theorem [H-Sottile]
The lifted formulation described above defines $X_w F_\bullet$ as a complete intersection.

$$\text{(Dimension } + \#\text{Equations } = \#\text{Variables)}$$

Corollary [H-Sottile]
The lifted formulation described above defines any instance of a Schubert problem as a square system.

$$\text{(#Equations } = \#\text{Variables)}$$

- There are similar results for Flag manifolds in type A.
Comparing formulations for a Schubert variety

Example \( X_{(5,9,10)} F_\bullet \subset \text{Gr}(3, 10), \quad X \longleftrightarrow \text{Gr}(3, 10). \)

- Determinantal formulation uses 45 degree 3 polynomials in 21 variables. (Reduces to 10 linearly independent polynomials)
- Primal-dual (square) formulation uses 21 bilinear polynomials in 39 variables.
- Lifted (square) formulation uses 6 bilinear polynomials in 24 variables. (Reduces to 5 relevant polynomials in 23 variables)
Comparing formulations for a Schubert problem

**Example** $\text{Gr}(3,10)$, \[ \mathcal{X} \leftrightarrow X^{(5,9,10)} F_1 \cap X^{(5,9,10)} F_2 \cap \cdots \cap X^{(7,9,10)} F_{15} \]

28,490 solutions

- Determinantal formulation uses 22 degree 3 polynomials in 15 variables.
- Primal-dual (square) formulation uses 21 bilinear polynomials and 12 degree 3 polynomials in 33 variables.
- Lifted (square) formulation uses 5 bilinear polynomials and 12 degree 3 polynomials in 17 variables.

- Solved and soft certified a random instance via lifted formulation
- $\sim 1.72$ gigaHertz-months: four 3.6 GHz processors, $\sim 3.6$ days
감사합니다

Thank you very much